# DESCRIPTION OF A CUBICAL INTEGRATOR.

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*qr*, *rk*, *kh*, *hq*, *qo*, *ok*, and *hn* are seven rods constituting a modified form of Peaucellier's original parallel motion, the fixed centres being *o* and *n*. The other extremities of the several rods move as shown by dotted lines, the point *r* moving in a straight line, perpendicular to *os*.

$FF'$ and *on* are the two principal pieces to the frame of the instrument. $FF'$ carries on its ends two broad wheels, *B* and $B'$, fluted in a manner similar to those on Coradi's planimeter, and thus, as they roll on the paper, the whole instrument moves in a straight line to any desired distance. This result could be accomplished in a number of ways, one being to make *B* and $B'$ narrow and compelling them to run in grooves. The device chosen seems, however, to be the best that can be adopted, as it has given remarkably accurate results in the planimeter mentioned.

*mm* is a disc pivoted on the frame at *o*. *T* is a small wheel on this disc, with finely divided spur teeth on its periphery engaging with corresponding teeth on *B*.

*ww* is a registering wheel like that on any planimeter.

At the point *o* we have the rods *oq* and *ok* pivoted together with the disc, but each moving independently.

The broken lines (– – – – – –) indicate the mid-position of the system of rods. It is evident that as the instrument rolls on the paper, the circumference of *B*, and hence of *T*, will move through a linear distance equal to that passed over by the instrument. As *ww* rests upon the disc, it is evident that as the instrument rolls along, a certain amount of motion will be communicated by the disc to *ww*. It is also evident that when the point *r* moves in a vertical direction the disc and wheels *B*, $B'$ must for that instant remain fixed. Hence, if *r* traces a curve and stops at the starting point, any vertical movement of the point *r* will not affect the reading of the wheel *ww*; for the sum of the decrements from such motion will equal the sum of the increments. The only permanent registration on *ww* will be the algebraic sum of what motion may be communicated to it by the disc *mm* as it revolves back and forth, and that disc can get motion only when *B* rolls on the paper and *r* remains (it may be for an instant only) at a constant distance above *os*.

Let the radius of *T* be *u*.

Also let *y* be the perpendicular distance of *r* above *os*, and let *dx* be the velocity, or rate, of *r* in the direction *os*.

Then $T$ will receive an angular motion,

$$d\theta = \frac{dx}{w}.$$

Draw $\rho$ and $\rho_1$. Let the angle $qbo = \omega$.

The plane $mm$ passes under $ww$ at a rate $\rho_1 d\theta$, which, for an infinitesimal movement, may be represented by $bf$ perpendicular to $\rho_1$.

From the figure it is evident that

$$fbw = qbo = \omega = 90^\circ - fbq.$$

Resolving $fb$ along $bw$, we get

$$fb \cos \omega = \rho_1 d\theta \cos \omega. \tag{1}$$

From the triangle $oqb$ we get

$$c^2 = e^2 + \rho_1^2 - 2e\rho_1 \cos \omega; \tag{2}$$

and from $obr$

$$\begin{aligned} \rho^2 &= \rho_1^2 + d^2 - 2\rho_1 d \cos (180^\circ - \omega) \\ &= \rho_1 + d^2 + 2\rho_1 d \cos \omega. \end{aligned} \tag{3}$$

From (2) and (3)

$$\cos \omega = \frac{\rho^2 - c^2 - d^2 + e^2}{2\rho_1 (e + d)}. \tag{4}$$

Substituting from (4) in (1), we get as the increment imparted to $ww$,

$$\frac{\rho^2 - c^2 - d^2 + e^2}{2(e + d)} d\theta.$$

In integrating between the limits of $\theta$ due to one complete oscillation of $mm$ as $r$ describes the curve $A$, the terms containing the constants will disappear, because $\theta$ after reaching its maximum returns to its initial value.

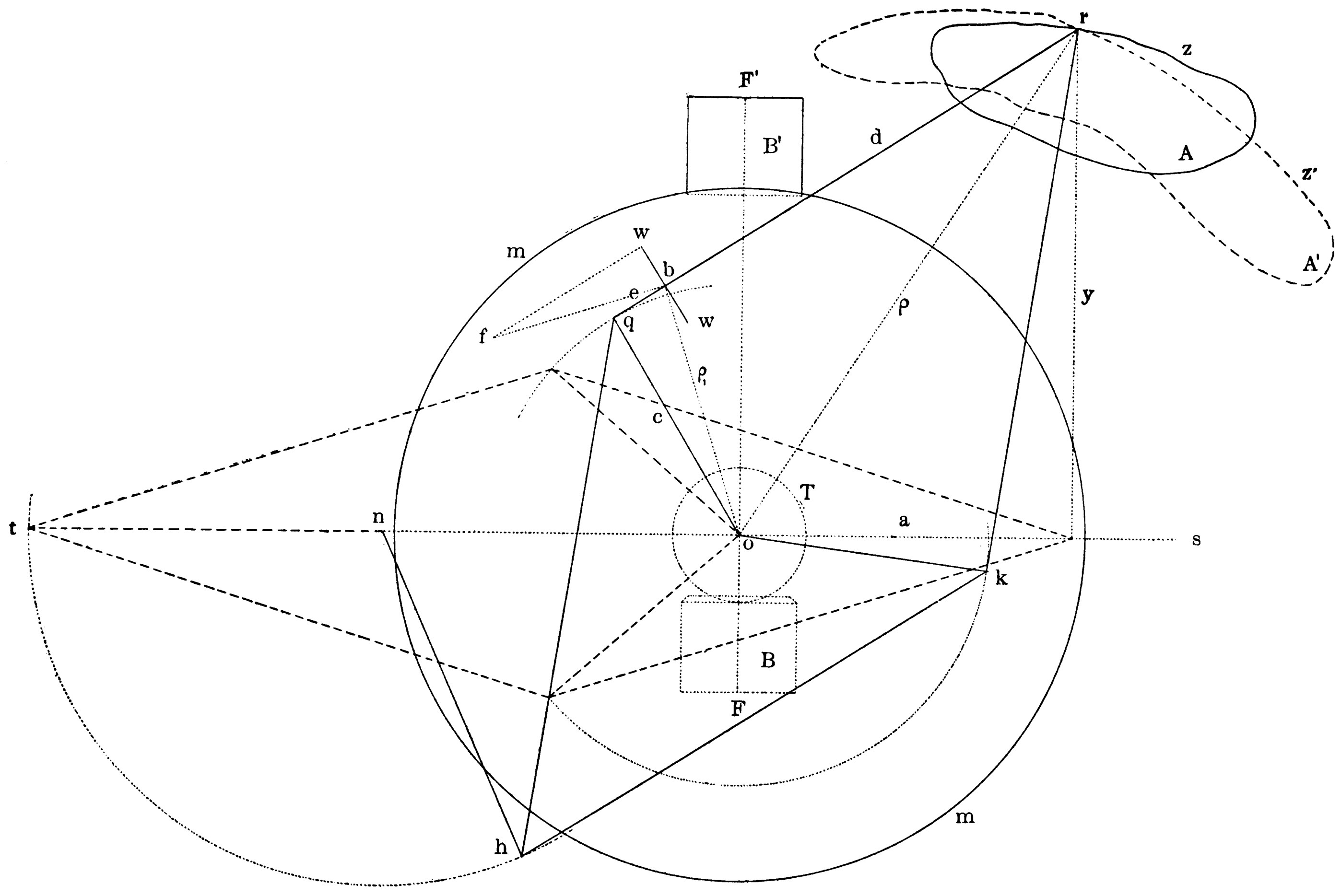
Hence the amount registered in an oscillation will be equal to

$$\frac{1}{2(e + d)} \int \rho^2 d\theta, \tag{5}$$

taken between proper limits, and will be confined to this value. This is evidently an area, but it is not that of $A$. It represents the area $A'$ which is enclosed by a curve gotten in this way: any point $z'$ is subject to the conditions that $z'o$ and the angle $z'or$ are, respectively, the values of $\rho$ and the angular motion of $mm$ when $r$ arrives at $z$.

It will be seen that (5) is independent of the position of $ww$ on $qr$ and of the length $oq$. In fact, this is the proof for an ordinary planimeter $oqr$ acting on $A'$. Now $\rho^2 = a^2 + y^2$; whence substituting in (5) we get

$$\frac{1}{2(e + d)} \int (a^2 + y^2)\, d\theta; \tag{6}$$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but $d\theta = \frac{dx}{u}$; whence (6) becomes

$$\frac{1}{2u(e+d)}\int(a^2 + y^2)\,dx. \tag{7}$$

When the instrument has completed its forward and backward trip, as much will have been additive in the above integral due to $a^2$ as is subtractive; hence the definite value of (7) after $r$ has finished the circuit will be due only to

$$\frac{1}{2u(e+d)}\int y^2 dx.$$

Hence $ww$ registers an amount

$$= \frac{\text{volume generated by revolution of } A \text{ about } os}{2u(e+d)\pi},$$

or

$$V = 2\pi u(e+d) \times (\text{amount registered on } ww),$$

in which the parenthesis depends on the diameter of $ww$ and the number of turns it makes. If $e$ and $u$ are in inches, the parenthesis will also have to be in inches and $V$ will be in cubic inches.

This proves, then, that when $r$ has gone entirely around the curve enclosing $A$ an amount has been registered on the measuring wheel $ww$ equal to the volume that would be generated by $A$ revolving about $os$ as an axis.

Such an instrument would be useful in getting the cubical contents, and hence the weight, of certain parts of machinery, all machinery in fact that has been turned up in lathes, or cast symmetrically with respect to some axis. In most of these cases the lower part of the curve would coincide with $os$.

It could also be used, supposing $oqr$ to be made detachable, to find the position of the centre of gravity of $A$. Thus: find the volume as shown; then detach $oqr$ and find the area; from these two the position of the centre of gravity above $os$ is determined by applying Guldin's theorem. Turn the instrument quarter way round, and we find the distance of the centre of gravity from an axis at right angles to $os$. These two distances fix its position.

I have shown $y$ as falling a little inside the disc $mm$. In a practical instrument the disc should be small enough to allow this so that $r$ could, if necessary move down to, or below, $os$.